JOURNAL OF ALGEBRA 56, 282–286 (1979)

A Question Concerning CZ-groups

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Communicated by Graham Higman

Received June 14, 1978

1. Introduction

CZ-groups were introduced by Kaplansky [2] as an abstraction of linear groups. A CZ-group is a group G which carries a topology satisfying the minimal condition for closed sets such that one-element subsets are closed and for each $a \in G$ the following maps from G to G are continuous: $x \mapsto x^{-1}$, $x \mapsto ax$, $x \mapsto xa$ and $x \mapsto x^{-1}ax$. Such a topology is called a CZ-topology. It is a familiar result that every finite extension of a linear group is itself linear: see Lemma 2.3 of Wehrfritz [3]. Question 9 of [3] asks if every finite extension of a CZ-group is a CZ-group. The object of this note is to answer the question negatively: a group G will be constructed which is not a CZ-group but which has a CZ-group H as a normal subgroup of finite index.

In [1] there was defined for any group a topology called the "verbal topology", closely related to the idea of a CZ-topology. The question was raised whether the minimal condition for closed sets in the verbal topology is a property preserved under finite extensions. Again the answer is negative: the same groups G and H suffice to show this. But in the interests of brevity we omit the proof.

2. The Group G

Let F be the free group on a countably infinite set of generators $\{x_0, x_1, ...\}$. Let $F_0 = F$ and, for $i \geqslant 0$, $F_{i+1} = F_i^2[F_i, F]$. Thus F_{i+1} is the smallest normal subgroup of F contained in F_i such that F_i/F_{i+1} is an elementary abelian 2-group and is central in F/F_{i+1}.

Clearly F/F_1 has a basis consisting of the elements $x_0F_1, x_1F_1, ...$.

It follows that F_1/F_2 is generated by the elements $x_i^2F_2$ and $[x_i, x_j]F_2$ (where $i > j$). Any element of F which belongs to F_2 is a law of the dihedral group of order 8. Consideration of this group shows that the elements $x_i^2F_2$ and $[x_i, x_j]F_2$ (where $i > j$) form a basis for F_1/F_2 .

0021-8693/79/010282–05$02.00/0

It follows that F_2/F_3 is generated by the elements $x_i^4F_3$, $[x_i, x_j]^2F_3$, $[x_i^2, x_k]F_3$ and $[x_i, x_j, x_k]F_3$, where $i, j, k \geqslant 0$. Now

$$[x_i^2, x_k]F_3 = [x_i, x_k]^2[x_i, x_k, x_i]F_3$$

and

$$[x_j, x_k, x_i][x_k, x_i, x_j]F_3 = [x_i, x_j, x_k]F_3 .$$

Hence F_2/F_3 is generated by the elements $x_i^4F_3$, $[x_i, x_j]^2F_3$ (where $i > j$) and $[x_i, x_j, x_k]F_3$ (where $i > j$ and $k \geqslant j$). Consideration of the dihedral group of order 16 shows that these elements form a basis for F_2/F_3. Furthermore, it is straightforward to verify that F_1/F_3 is abelian.

Let N/F_3 be the subgroup of F_2/F_3 generated by those elements

$$[x_i, x_0, x_j][x_j, x_0, x_i]F_3$$

where $i, j \neq 0$ and $|i - j| \neq 1$. Let $G = F/N$. We write $G_1 = F_1/N$, $G_2 = F_2/N$ and $g_i = x_iN$ for all i. Thus G/G_1, G_1/G_2 and G_2 are elementary abelian 2-groups such that $[G, G] \leqslant G_1$, $[G_1, G] \leqslant G_2$, $[G_2, G] = \{1\}$ and $[G_1, G_1] = \{1\}$. In particular, G is nilpotent of class 3.

G/G_1 has a basis consisting of the elements g_iG_1 while G_1/G_2 has a basis consisting of the elements $g_i^2G_2$ and $[g_i, g_j]G_2$ (where $i > j$). Finally, G_2 has a basis consisting of the elements g_i^4, $[g_i, g_j]^2$ (where $i > j$), and certain elements $[g_i, g_j, g_k]$. To examine G_2 further let us call elements of the form $[g_i, g_j, g_k]$, with $i > j$, *triple commutators* and let us say that triple commutators $[g_i, g_j, g_k]$ and $[g_{i'}, g_{j'}, g_{k'}]$ have the same *type* if (i, j, k) is a permutation of (i', j', k'). By the choice of N, a triple commutator $[g_i, g_j, g_k]$ is trivial if and only if $k = 0$, $j > 0$ and $i > j + 1$. Also, a non-trivial triple commutator $[g_i, g_j, g_k]$ does not belong to the subgroup generated by the triple commutators of type different from $[g_i, g_j, g_k]$.

We now show that G is not a CZ-group. Suppose otherwise. Then the maps $x \mapsto [x, g_j]$ are continuous. Hence for each j the set

$$S_j = \{x \in G: [x, g_j, g_0] = 1\}$$

is closed. Furthermore, for all $j \geqslant 1$,

$$g_{j+2} \in (S_1 \cap \cdots \cap S_j)\backslash(S_1 \cap \cdots \cap S_{j+1}).$$

Hence

$$S_1 \supset (S_1 \cap S_2) \supset (S_1 \cap S_2 \cap S_3) \supset \cdots$$

is an infinite properly descending chain of closed subsets of G. This is the required contradiction.

3. The Group H

Let H/G_1 be the subgroup of G/G_1 generated by the elements g_iG_1 where $i \neq 0$. Thus H is a normal subgroup of G of index 2. We shall prove that H is a CZ-group.

Let $\mathscr{X}$ be the set of those subsets of H of the following forms:

$$H,\ hG_1\,,\ hG_2\,,\ \{h\},\ \varnothing\ \ (h \in H).$$

Let $\mathscr{T}$ be the topology on H in which $\mathscr{X}$ is a sub-basis for the closed sets. Thus every closed set is an intersection of finite unions of elements of $\mathscr{X}$. (It follows, in fact, that every closed set is a finite union of elements of $\mathscr{X}$.) To prove that H is a CZ-group we shall show that $\mathscr{T}$ is a CZ-topology. (More complicated calculations show that $\mathscr{T}$ is the verbal topology on H.) Clearly $\mathscr{X}$ is closed under finite intersections and satisfies the minimal condition. Hence, by Lemma 3.2 of [1], the minimal condition is satisfied for arbitrary closed sets. Clearly one-element subsets are closed and the maps $x \mapsto x^{-1}$, $x \mapsto ax$ and $x \mapsto xa$ are continuous. It only remains to prove that the maps $x \mapsto x^{-1}ax$ are continuous. For this it is enough to prove that each map $x \mapsto [x, a]$ is continuous.

Let $a \in H$ and $X \in \mathscr{X}$. Then it suffices to show that the set

$$T = \{x \in H\colon [x, a] \in X\}$$

is a finite union of elements of $\mathscr{X}$. This is obvious if X has the form H or $\varnothing$. Suppose then that $X = hG_\lambda$ $(1 \leqslant \lambda \leqslant 3)$ where we take $G_3 = \{1\}$. Clearly we may assume that T is non-empty: say $b \in T$. Then $T = Sb$ where

$$S = \{x \in H\colon [x, a] \in G_\lambda\}.$$

Thus it is enough to show that S is a finite union of elements of $\mathscr{X}$. If $\lambda = 1$ we have $S = H$. Thus it remains to consider the cases $\lambda = 2$ and $\lambda = 3$.

Suppose first that $\lambda = 2$, i.e.

$$S = \{x \in H\colon [x, a] \in G_2\}.$$

If $a \in G_1$ then $S = H$. Thus we assume $a \in H\backslash G_1$. Clearly $G_1 \cup aG_1 \subsetneq S$. Let $x \in S\backslash G_1$. Then we can write $xG_1 = \xi_1 \cdots \xi_k G_1$ and $aG_1 = \alpha_1 \cdots \alpha_l G_1$ where the ξ_i are distinct elements from $\{g_1\,, g_2\,, \ldots\}$, as are the α_j, and where $k, l \geqslant 1$. Since $[x, a] \in G_2$ we have $\prod[\xi_i\,, \alpha_j] \in G_2$. From this it follows that $\{\xi_1\,, \ldots, \xi_k\}$ and $\{\alpha_1\,, \ldots, \alpha_l\}$ are equal, so $xG_1 = aG_1$. Hence $S = G_1 \cup aG_1$.

Finally, suppose that $\lambda = 3$, i.e.

$$S = \{x \in H\colon [x, a] = 1\}.$$

To deal with this case we need a lemma.

LEMMA. *Suppose that* $[c, d] = 1$ *where* $c \in G_1 \backslash G_2$ *and* $d \in H \backslash G_1$. *Then* $cG_2 = d^2 G_2$. *Furthermore, if also* $d' \in H \backslash G_1$ *and* $[c, d'] = 1$ *then* $dG_1 = d'G_1$.

Proof. We can write $dG_1 = \delta_1 \cdots \delta_m G_1$ where the δ_j are distinct elements of $\{g_1, g_2, \ldots\}$ and $m \geqslant 1$. We shall show that $c \notin [G, G]G_2$.

Suppose otherwise. Then we can write

$$cG_2 = [\beta_1, \gamma_1] \cdots [\beta_l, \gamma_l] G_2$$

where the $[\beta_i, \gamma_i]$ are distinct elements of the form $[g_s, g_t]$ with $s > t \geqslant 0$ and where $l \geqslant 1$. Since $[c, d] = 1$ we have

$$\prod [\beta_i, \gamma_i, \delta_j] = 1.$$

Suppose first that $\beta_i = \delta_j$ for some i, j. Then the corresponding factor $[\beta_i, \gamma_i, \delta_j]$ is non-trivial, because $\delta_j \neq g_0$, and is independent of the other factors, because they have different types. This is impossible. Hence $\beta_i \neq \delta_j$ for all i, j. Similarly, $\gamma_i \neq \delta_j$ for all i, j. It now follows that the factor $[\beta_1, \gamma_1, \delta_1]$ is independent of the other factors, because they have different types. This is a contradiction. Hence $c \notin [G, G]G_2$.

Thus we can write

$$c[G, G]G_2 = \alpha_1^2 \cdots \alpha_k^2 [G, G] G_2$$

where the α_i are distinct elements of $\{g_0, g_1, \ldots\}$ and $k \geqslant 1$. Since $[c, d] = 1$ we have

$$\prod [\alpha_i^2, \delta_j] \in [G, G, G],$$

giving

$$\prod [\alpha_i, \delta_j]^2 \in [G, G, G].$$

Therefore $\prod [\alpha_i, \delta_j]^2 = 1$. From this it follows that $\{\alpha_1, \ldots, \alpha_k\}$ and $\{\delta_1, \ldots, \delta_m\}$ are equal, so $dG_1 = \alpha_1 \cdots \alpha_k G_1$. Hence $d^{-2}c \in [G, G]G_2$. If we had $d^{-2}c \notin G_2$ then, since $[d^{-2}c, d] = 1$, the argument used in the first part of the proof would yield a contradiction. Hence $d^{-2}c \in G_2$ and so $cG_2 = d^2 G_2$.

For the last statement of the lemma note that, since $dG_1 = \alpha_1 \cdots \alpha_k G_1$, dG_1 is determined uniquely by c.

Returning to the main proof, if $a \in G_2$ then $S = H$. So we assume $a \notin G_2$. Suppose first that $a \in G_1 \backslash G_2$. Then clearly $G_1 \subseteq S$. If $S = G_1$ we have finished. So assume that there exists $x \in S \backslash G_1$. Then, by the Lemma, xG_1 is uniquely determined by a. Hence, for some b, $S = G_1 \cup bG_1$. It remains only to consider the case where $a \in H \backslash G_1$. By the result for $\lambda = 2$, we have $S \subseteq G_1 \cup aG_1$. But it is easy to see that $S \cap aG_1 = a(S \cap G_1)$. Hence $S = (S \cap G_1) \cup a(S \cap G_1)$.

Clearly $G_2 \cup a^2G_2 \subseteq S \cap G_1$. Let $x \in (S \cap G_1)\backslash G_2$. Then, by the Lemma, $xG_2 = a^2G_2$. Hence $S \cap G_1 = G_2 \cup a^2G_2$ giving

$$S = G_2 \cup aG_2 \cup a^2G_2 \cup a^3G_2 .$$

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Printed in Belgium